Lecture 16: More on matrix multiplication

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remember

- matrix multiplication is not commutative
- in general, no cancellation is possible

ie.

numbers	matrices
$(a+b)\cdot(a-b)$	$(A+B)\cdot(B-A)$
$=a^2-b^2$	$=AA-AB+BA-BB$
	$=A^2-AB+BA-B^2$

Cayley-Hamilton Theorem

Let $A \in \mathrm{M}_{22}(\mathbb{R})$ and consider:

$$p(x)=x^2-tr(A)\cdot x+\det(A)$$

Then,

$$p(A)=0$$

14.5 Block Multiplication

Treat sub-matrices as numbers:

$$A=\left[\begin{array}{ccc|cc}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\\hline 0&0&0&0&1\\0&0&0&0&0\end{array}\right]=\left[\begin{array}{c|c}B&0\\\hline 0&C\end{array}\right] \qquad \text{with } B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \text{ and } C=\begin{bmatrix}0&1\\0&0\end{bmatrix}$$

$$A^2=\begin{bmatrix}B&0\\0&C\end{bmatrix}\cdot\begin{bmatrix}B&0\\0&C\end{bmatrix}=\begin{bmatrix}B^2&0\\0&C^2\end{bmatrix}=\begin{bmatrix}B&0\\0&0\end{bmatrix}$$

$$A^{100}=\begin{bmatrix}B&0\\0&0\end{bmatrix}$$

14.6 Matrices and linear systems

The following are equivalent:

a) linear system:

$$\begin{aligned}x+2x_2+3x_3&=4\\x_1-x_2+\ \ x_3&=2\\x_2-3x_3&=0\end{aligned}$$

b) matrix equation:

$$\underbrace{\begin{bmatrix}1&2&3\\1&-1&1\\0&1&-3\end{bmatrix}}_{A}\cdot\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{x}=\underbrace{\begin{bmatrix}4\\2\\0\end{bmatrix}}_{b}$$

They can both be solved with Gaussian elimination.

Let's go further with b):

$$\underbrace{\begin{bmatrix}1&2&3\\1&-1&1\\0&1&-3\end{bmatrix}}_{A}\cdot\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{X}=[C_1\ \ C_2\ \ C_3]\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$with\ \ C_1=\begin{bmatrix}1\\1\\0\end{bmatrix}, C_2=\begin{bmatrix}2\\-1\\1\end{bmatrix}, C_3=\begin{bmatrix}3\\1\\-3\end{bmatrix}$$

$$=x_1C_1+x_2C_2+x_3C_3$$

(*linear combinations of the columns of A*)

Hence, Ax is a linear combination of the columns of A.
This allows us to say:

(i) $Ax = b$ is consistent $<=>$ b is a linear combination of the columns of A

(ii) $Ax = 0$ has a unique solution (x=0)
$<=>$ the columns of A are linearly independent
$<=> rank(A) = \# \, columns \, of \, A$

4.7 Definition of column space

Let $A = [C_1 \, C_2 \dots C_n]$. Then, $col(A) := span\{C_1, C_2, \dots, C_n\}$.

Check for linear independence

1) Write vectors in columns of A
2) Perform Gaussian elimination
3) Count leading ones: if there's a leading one in each column:
   ⇒ linearly independent

   otherwise:
   ⇒ linearly dependent

$$\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\} \quad \begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\\1 & 3 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\\0 & 1 & 1\end{bmatrix} \sim \underbrace{\begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}}_{REF}$$

$rank(A) < \# \, columns \, of \, A$
⇒ linearly dependent

We can write $Col(A) = \{Ax \mid x \in \mathbb{R}^n\}$.

- this is the **image** of A, or $im(A)$

15 Further spaces associated to matrices

15.1 Definitions

(i) Let $A = \begin{bmatrix}r_1\\ \vdots \\ r_m\end{bmatrix}$, that is the matrix with rows $r_1, \dots, r_m$.
Then, $Row(A) := \mathrm{span}\{r_1, \dots, r_m\}$ is the **row space** of A. Typically, we transpose the rows to obtain column vectors in $\mathbb{R}^n$.

(ii) Let $A \in M_{mn}(\mathbb{R})$. Then, $Null(A) := \{x \in \mathbb{R}^n \mid Ax = 0\}$ is the **nullspace** or **kernel** of A (ker(A)).

→ Both are subspaces of $\mathbb{R}^n$!

Example:

$$A = \begin{bmatrix}1 & 2 & 3\\0 & 0 & 0\end{bmatrix} \qquad Col(A) = span\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}\}$$

$$= \left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}t\\0\end{pmatrix} \middle| \, t \in \mathbb{R}\right\}$$

$$Row(A) = span\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right\} = span\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix}t\\2t\\3t\end{pmatrix} \middle| \, t \in \mathbb{R}\right\}$$

unnecessary

$$Null(A) = \left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \middle| \begin{bmatrix}1 & 2 & 3\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}\right\}$$

augmented matrix

$\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\0 & 0 & 0 & 0\end{array}\right]$

RREF

$$= \left\{\begin{pmatrix}-2s-3t\\s\\t\end{pmatrix} \middle| \, s, t \in \mathbb{R}\right\}$$

$$= \left\{s \cdot \begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}-3\\0\\1\end{pmatrix} \middle| \, s, t \in \mathbb{R}\right\}$$

$$= span\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix},\begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$$